Models with continuous symmetry :

O(N) model in d - dimensions

O(N) model (N>1) has two fundamental differences from the Ising model : (i) The ordered phase (i.e. spontaneous sym. breaking phase) has massless excitations (Goldstone modes e.g. spin-waves or phonons).

(ii) Compared to a system with discrete sym., fluctuations cost parametrically less energy resulting in a lower critical dimension $d_L = 2$ (compared to $d_L = 1$ for the Ising model). These two facts are not unrelated - close to $d = d_L$ Goldstone modes start interacting strongly and eventually destroy long-range order below d_L. As we will see, the case $N = 2$, $d = d_L = 2$ is quite special in many ways.

Goldstone modes:

Let's write the $O(N)$ model Hamiltonian as

$$H = -J \sum_{\langle r r' \rangle} \vec{n}(\vec{r})\, \vec{n}(\vec{r}\,') \quad \text{where } \vec{n} \text{ is an}$$

N-component unit vector and $r, r' \in$ a d-dimensional lattice.

Let's assume that sym. is spontaneously broken at low enough temperature and parametrize $\vec{s}$ in the SSB phase as $\vec{n} = \left(\sqrt{1-\vec{\sigma}^2}, \vec{\sigma}\right)$ where the first component $\left(= \sqrt{1-\vec{\sigma}^2}\right)$ lies along the direction along which the SSB occurs, and $\vec{\sigma}$ therefore is $n-1$ component vector with $|\vec{\sigma}| \ll 1$ at low temperatures. The SSB phase has $O(N-1)$ symmetry.

In the spirit of our earlier discussion, let's write a coarse-grained Hamiltonian/action

$$S = \frac{1}{2g} \int d^d x \left(\partial_\mu \vec{n}\right)^2$$

where g plays the role of temperature.

$$= \frac{1}{2g}\int d^dx \quad \left[\partial_\mu\left(\sqrt{1-\vec{\sigma}^2}\right)\right]^2 + (\partial_\mu\vec{\sigma})^2 \Big]$$

$$= \frac{1}{2g}\int d^dx \left[\frac{(\vec{\sigma}\cdot\partial_\mu\vec{\sigma})^2}{1-\vec{\sigma}^2} + (\partial_\mu\vec{\sigma})^2\right]$$

Rescaling $\frac{\vec{\sigma}^2}{g} \to \vec{\sigma}^2$ so that the coefficient of $\frac{(\partial_\mu\vec{\sigma})^2}{2}$ is unity,

$$S = \int d^dx \left[\frac{(\partial_\mu\vec{\sigma})^2}{2} + \frac{1}{2}g\frac{(\vec{\sigma}\cdot\partial_\mu\vec{\sigma})^2}{1-g\vec{\sigma}^2}\right]$$

Taylor expanding, one finds

$$S = \int d^dx \Big[\frac{(\partial_\mu\vec{\sigma})^2}{2} + \frac{g}{2}(\vec{\sigma}\cdot\partial_\mu\vec{\sigma})^2$$
$$+ \frac{1}{2}g^2\vec{\sigma}^2(\vec{\sigma}\cdot\partial_\mu\vec{\sigma})^2\Big]$$

There is quite a bit of physics in this action;

(a) $\vec{\sigma}$ are precisely the Goldstone modes. There are $N-1$ of them. # of generators of $O(N)$ $-$ # of generators of $O(N-1)$ $= N\frac{(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1.$

(b) There is no mass term in the action for goldstone modes. A term such as $\vec{h}\cdot\vec{n}$ will indeed lead to a mass for $\vec{\sigma}$ but such a term breaks the O(N) sym. explicitly.

(c) Goldstone modes interact via interactions that involve derivatives $\sim g\,(\vec{\sigma}\cdot\partial_\mu\vec{\sigma})^2$, unlike the φ^4 action of the Ising model. Due to derivatives, the interactions are weak.

All in all, if one neglects interactions between the Goldstone modes, then the low energy theory is simply $N-1$ massless fields described by a purely Gaussian theory $\sim \dfrac{(\partial_\mu\vec{\sigma})^2}{2}$.

Now one may ask: when do Goldstone modes start interacting strongly?

Scaling dim. of $\vec{\sigma}$ in the Goldstone phase

$$= \frac{d-2}{2}$$

$\Rightarrow$ In this Gaussian theory,

$$\left(\vec{\sigma} \cdot \partial_\mu \vec{\sigma} \right)^2 (x) \quad \left(\vec{\sigma} \cdot \partial_\mu \vec{\sigma} \right)^2 (0)$$

$$\sim \quad \frac{1}{x^{4(d-2)+4}} \quad \sim \quad \frac{1}{x^{2[2d-2]}}$$

$\Rightarrow$ Scaling dim. x_g of the interaction term

$$\left(\vec{\sigma} \cdot \partial_\mu \vec{\sigma} \right)^2 = 2d - 2$$

For this term to be irrelevant, we require.

$$y_g = d - x_g < 0$$

$$\Rightarrow \quad \boxed{d > 2} \; .$$

Therefore, Goldstone modes do not interact strongly in $d > 2$, which suggests that the lower critical dimension $d_L = 2$.

There are a few other essentially equivalent ways to arrive at this same conclusion:

(a) Recall that the order parameter is given by

$$\left\langle \sqrt{1-\vec{\sigma}^{\,2}} \right\rangle \simeq 1 - \frac{\langle \vec{\sigma}^{\,2}_{(\vec{r})} \rangle}{2} \simeq 1 - (n-1) \int \frac{d^d k}{k^2}$$

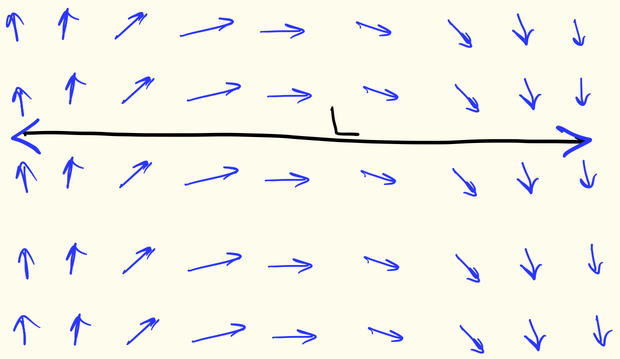
where we have used the fact that $\langle \vec{\sigma}(\vec{r}) \cdot \vec{\sigma}(0) \rangle$

$$\sim (n-1) \int \frac{d^d k}{k^2} e^{ik \cdot r} .$$

Since the integral $\int_0^\Lambda \frac{d^d k}{k^2}$ diverges in $d \leq 2$

and converges in $d > 2$ $\Rightarrow$ the ordering is stable against fluctuations in $d > 2$ at low enough temperatures.

Equivalently, consider the following fluctuation around the ground state:



The key difference with the Ising model is that due to continuous sym, one can create a configuration which has lower energy than a domain wall.

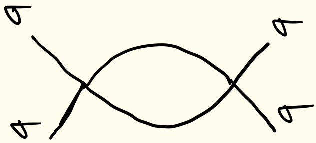
$$\text{Energy cost} \sim \int d^d x \, (\nabla \theta)^2 \sim \frac{L^d}{L^2} \sim L^{d-2}$$

Entropy cost $\sim$ log (# of locations where the config. points along $\longrightarrow$) $\sim \log(L)$.

Free energy cost $\sim L^{d-2} - T \log(L) \gg 0$ when $d > 2 \Rightarrow$ fluctuations suppressed in $d > 2$.

Contrast this with a discrete sym model such as the Ising model where due to discreteness, energy $\sim L^{d-1}$ leading to lower critical dimension $d_L = 1$.

(b) Another way to arrive at the same conclusion is to consider the lowest order correction to the interaction vertex:



$$\sim k^4 \int \frac{d^d p}{(p+k)^2 \, k^2} \sim k^d$$

due to derivative interaction $(\pi \cdot \partial_\mu \pi)^2$.

Therefore, fractional change in the interaction vertex

$$\sim \frac{k^d}{k^2} \sim k^{d-2} \quad \text{in } d > 2$$

$$\sim \log(k) \quad \text{in } d = 2.$$

$\Rightarrow$ The relative change in the interaction is negligible in $d > 2$ at low energies.